UNIT - I

Def: velocity

The rate of displacement that is the rate of change of position is called the velocity of the particle. Velocity is denoted by $\vec{v}$

$$\vec{v} = \frac{d\bar{r}}{dt}.$$

Relative velocity:

Let A and P be two moving points. Then $\bar{AP}$ is the position vector of P with reference to A. ie), AP specifies the position of P relative to A.

So $\frac{d}{dt}(\bar{AP})$ is velocity of P relative to A.

If A and P are two moving points. then $\frac{d}{dt}(\bar{AP})$ is the velocity of P relative to A.

let, $\frac{d}{dt}(\bar{AP}) = \frac{d}{dt}(\bar{OP} - \bar{OA}) = \bar{VP} - \bar{VA}$

where $\bar{VA}$ and $\bar{VP}$ are the velocities of A and P

Thus the velocity of a point P relative to another point A. The vector sum of the velocity of P and the reversed velocity of A.

Motion of P relative to A:

This motion is equivalent to the situation with A is at rest and P moves with a velocity

$\vec{v}_1 - \vec{v}_A$ obtaining this situation is said to reduce A to rest assuming P to move with a velocity which is the resultant of

(i) Velocity of P and

(ii) Reversed velocity of A.

Def : Acceleration

Acceleration of a particle is the time-rate of change of its velocity. That is if $\vec{v}$ is its velocity at time t. Then its acceleration $\bar{a}$ at time t is $\frac{d\vec{v}}{dt}$

Rectilinear motion :

When a particle moves along a straight line the motion of the particle is said to be rectilinear.

Rectilinear motion with a Constant acceleration :

Book work :

Let a particle moves along a straight line with a constant acceleration then to show that

$v = u + at$

$s = ut + \frac{1}{2}at^2$

$v^2 = u^2 + 2as$

where u is the initial velocity of the particle and

a) v is the velocity of the particle at time t.

b) s is the distance of the particle at time t from a choosen fixed point on the line.

Soln

(i) Now the Scalar acceleration is 'a'

$$\frac{d^2s}{dt^2} = a$$

$$\frac{d}{dt}\left(\frac{ds}{dt}\right) = a$$

$$\frac{d}{dt}(v) = a$$

$$dv = a\,dt$$

Sing with r. to 't'

$$\int dv = a\int dt$$

$$V = at + A.$$

when $t=0$, $v=u$ this given that $A = u$, So

$$\boxed{v = u+at}$$

(ii) $$\frac{ds}{dt} = u+at$$

$$ds = u\,dt + at\,dt$$

$$\int ds = u\int dt + a\int t\,dt$$

$$S = ut + \tfrac{1}{2}at^2 + B$$

But when $t=0$, $S=0$ this given that

$$0 = 0+0+B$$

$$B = 0$$

$$\boxed{S = ut + \tfrac{1}{2}at^2}$$

(iii) W.K.T acceleration can be written as

$$v^2 = (u+at)^2$$

$$= u^2 + a^2t^2 + 2uat$$

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$$v^2 = u^2 + 2a\left(\frac{at^2}{2} + ut\right)$$

$$= u^2 + 2a\left(ut + \tfrac{1}{2}at^2\right)$$

$$\boxed{v^2 = u^2 + 2as}$$

Problems :

1. If a point moves in a straight line with uniform acceleration and covers successive equal distances in times t_1, t_2, t_3 then S.T

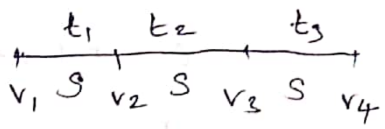
$$\frac{1}{t_1} - \frac{1}{t_2} + \frac{1}{t_3} = \frac{3}{t_1 + t_2 + t_3}.$$

Soln:

let s be the successive equal distances and v_1, v_2, v_3 the initial velocities for the successive distances and v_4 the given final velocity in the third distance. Since the acceleration is a constant the mean velocities in the three intervals and in the total interval are,

$$\frac{1}{2}(v_1+v_2),\ \tfrac{1}{2}(v_2+v_3),\ \tfrac{1}{2}(v_3+v_4),\ \tfrac{1}{2}(v_1+v_4)$$

$$\frac{s}{t_1} = \tfrac{1}{2}(v_1+v_2) \rightarrow (1)$$



$$\frac{s}{t_2} = \tfrac{1}{2}(v_2+v_3) \rightarrow (2)$$

$$\frac{s}{t_3} = \tfrac{1}{2}(v_3+v_4) \rightarrow (3)$$

$$\frac{3s}{t_1+t_2+t_3} = \tfrac{1}{2}(v_1+v_4) \rightarrow (4)$$

① − ② + ③

$$\frac{s}{t_1} - \frac{s}{t_2} + \frac{s}{t_3} = \frac{v_1 + v_2 - v_2 - v_3 + v_3 + v_4}{2}$$

$$s\left[\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_2}\right] = \frac{1}{2}(v_1 + v_4)$$

$$s\left[\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3}\right] = \frac{3s}{t_1+t_2+t_3}$$

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_2} = \frac{3}{t_1+t_2+t_3}$$

∴ Hence. proved.

2. A train moving at 30m/sec reduces its speed to 10m/sec in a distance of 240m at what distance will the train come to a stop? If the break power is increased by $12\frac{1}{2}$% .. S.T the train will stop in a total distance of 240m.

Soln:

In the first and second cases, let the retardation be $a\,m/sec^2$, $a'\,m/sec^2$. Now for the first phase of the first case we have

$u = 30$, $v = 10$, $s = 240$

So by

$v^2 = u^2 + 2as$. we get a from

$$100 = 900 + (-2a)(240)$$

$$-800 = -480a$$

$$a = \frac{800}{480}$$

$\boxed{a = 5/3}$

let S_1 be the distance travelled by the train in the second phase. In that phase the initial velocity is 10 and the final velocity is 0 and therefore

$$0 = 10^2 - 2aS_1$$

$$S_1 = \frac{100}{2a} = \frac{100 \times 3}{2 \times 5}$$

$$S_1 = 30\text{ m}$$

In the second case the retardation is

$$a' = a + a \times \frac{12\frac{1}{2}}{100}$$

$$= \frac{9a}{8} = 9/8 \times 5/3$$

$$a' = 15/8$$

If S_2 is the distance in this case.

$$0 = 30^2 - 2(15/8)S_2$$

$$\boxed{S_2 = 240}$$

3. The speed of a train increased at a constant rate α from '0' to v and then remains constant for an interval and finally decreases to '0' at a constant rated β. If S is the total distance described. P.T the total time T occupied is $T = S/v + \frac{v}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$

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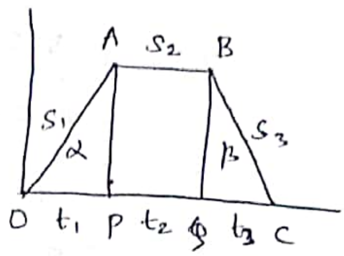
Soln:

let the speed of a train increased at a Constant rate α from 'o' to V and then remains for an interval and finally decreases to 'o' at a Constant rate β.



If s is the total distance described.

Let the durations of the three intervals be t_1, t_2, t_3 and the Constant velocity be V.

At OA:

$$v = u + at$$

Initial velocity $u = 0$.

Final velocity $v = V$

$$a = \alpha$$

$$t = t_1$$

$$V = 0 + \alpha t_1$$

$$\boxed{t_1 = V/\alpha} \longrightarrow (1)$$

$$V^2 = u^2 + 2as$$

$$V^2 = 2\alpha S_1$$

$$\boxed{S_1 = V^2/2\alpha} \longrightarrow (2)$$

At AB:

$$\text{Speed} = \frac{\text{distance}}{\text{time}}$$

$$V = \frac{S_2}{t_2}$$

$$\boxed{t_2 = \frac{S_2}{V}} \longrightarrow (3)$$

At BC:

Initial velocity $u = V$

$v = 0$

$a = -\beta$

$t = t_3$

$S = S_3$

$v = u + at$

$0 = V - \beta t_3$

$$\boxed{t_3 = V/\beta} \longrightarrow (4)$$

$v^2 = u^2 + 2as$

$0 = V^2 - 2\beta S_3$

$$\boxed{S_3 = \frac{V^2}{2\beta}} \longrightarrow (5)$$

(1) + (3) + (4)

$T = t_1 + t_2 + t_3$

$= \frac{V}{\alpha} + \frac{S_2}{V} + \frac{V}{\beta}$

$$T = V\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + \frac{S_2}{V} \longrightarrow (6)$$

$S = S_1 + S_2 + S_3$

$= \frac{V^2}{2\alpha} + S_2 + \frac{V^2}{2\beta}$

$S_2 = S - \frac{V^2}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)$

$$T = V\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + \frac{S - V^2/2 \cdot (1/\alpha + 1/\beta)}{V}$$

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$$T = \frac{v^2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) + \frac{2S - v^2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)}{2}}{v}$$

$$= \frac{2v^2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right) + 2S - v^2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)}{2v}$$

$$= \frac{2S + v^2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)(2-1)}{2v}.$$

$$T = \frac{2S}{2v} + \frac{v^2\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)}{2v}.$$

$$T = \frac{S}{v} + \frac{v}{2}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)$$

Hence, proved.

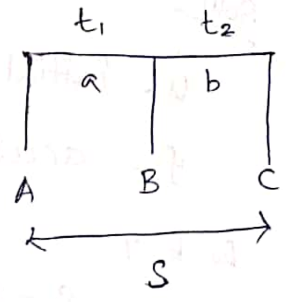
4. A train moves in a straight line with a uniform acceleration and described distances 'a' and 'b' in successive intervals of durations t_1 and t_2. S.T its acceleration is $f = \frac{2(bt_1 - at_2)}{t_1 t_2 (t_1 + t_2)}$

Soln:

u - initial velocity

v - final velocity

f - Constant acceleration



W.K.T

Equation of motion

$$S = ut + \tfrac{1}{2}at^2$$

At AB:

$$a = ut_1 + \tfrac{1}{2}ft_1^2 \longrightarrow (1)$$

At AC:

$$a+b = u(t_1+t_2) + \frac{1}{2} f (t_1+t_2)^2 \longrightarrow (2)$$

$$(2) \times t_1 \Rightarrow (a+b)t_1 = u t_1 \cancel{(t_1+t_2)} + \frac{1}{2} f t_1 (t_1+t_2)^2$$

$$(1) \times (t_1+t_2) \Rightarrow a(t_1+t_2) = u t_1 \cancel{(t_1+t_2)} + \frac{1}{2} f t_1^2 (t_1+t_2)$$

$$(-) \qquad (-) \qquad (-)$$

$$(a+b)t_1 - a(t_1+t_2) = \frac{1}{2} f t_1 (t_1+t_2)^2 - \frac{1}{2} f t_1^2 (t_1+t_2)$$

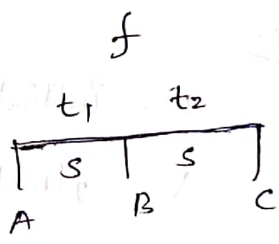
$$at_1 + bt_1 - at_1 - at_2 = \frac{1}{2} f t_1 (t_1+t_2)(t_1+t_2-t_1)$$

$$2(bt_1 - at_2) = f t_1 t_2 (t_1+t_2)$$

$$f = \frac{2(bt_1 - at_2)}{t_1 t_2 (t_1+t_2)}$$

5. A train moves in a straight line with a uniform acceleration and describes equal distance s in two successive interval of durations t_1 and t_2, S.T its acceleration is $f = \frac{2s(t_1 - t_2)}{t_1 t_2 (t_1+t_2)}$

Soln:



u - initial velocity

f - acceleration

W.K.T

$$s = ut + \frac{1}{2} at^2$$

At AB:

$$s = ut_1 + \frac{1}{2} f t_1^2 \longrightarrow (1)$$

At AC:

$$2s = u(t_1+t_2) + \frac{1}{2} f (t_1+t_2)^2 \longrightarrow (2')$$

① $\times (t_1+t_2) \Rightarrow s(t_1+t_2) = ut_1(t_1+t_2) + \frac{1}{2} f t_1^2 (t_1+t_2)$

② $\times\, t_1 \Rightarrow \underset{(-)}{2st_1} = \underset{(-)}{ut_1(t_1+t_2)} + \underset{(-)}{\frac{1}{2} f t_1 (t_1+t_2)}$

$$s(t_2-t_1) = \frac{1}{2} f t_1 (t_1+t_2)(t_1-(t_1+t_2))$$

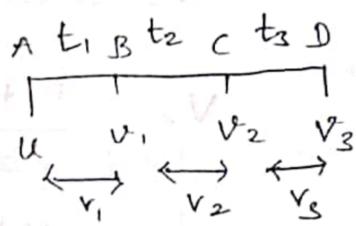
$$s(t_2-t_1) = \frac{1}{2} f t_1 (t_1+t_2)(-t_2)$$

$$2s(t_1-t_2) = f t_1 t_2 (t_1+t_2)$$

$$f = \frac{2s(t_1-t_2)}{t_1 t_2 (t_1+t_2)}.$$

6. A point moves with a uniform acceleration and v_1, v_2, v_3 denote its average velocities in three successive intervals of time t_1, t_2, t_3. P.T $v_1 - v_2 : v_2 - v_3 = t_1 + t_2 : t_2 + t_3$

Soln:



u – initial velocity

a – uniform acceleration

v_1, v_2, v_3 – velocities

t_1, t_2, t_3 – intervals.

$$v = u + at$$

At AB:

$t = t_1,\ u = u_1,\ v = v_1$

$$v_1 = u + at_1$$

At BC:

$t = t_2,\ u = v_1,\ v = v_2$

$v_2 = v_1 + at_2$

$v_2 = u + at_1 + at_2$

$v_2 = u + a(t_1 + t_2)$

At CD:

$t = t_3,\ u = v_2,\ v = v_3$

$v_3 = v_2 + at_3$

$= u + a(t_1 + t_2) + at_3$

$v_3 = u + a(t_1 + t_2 + t_3)$

V_1 = average velocity t_1

$$V_1 = \frac{u + v_1}{2} = \frac{u + u + at_1}{2}$$

$$V_1 = \frac{2u + at_1}{2} \longrightarrow 1$$

V_2 = average velocity t_2

$$V_2 = \frac{v_1 + v_2}{2}$$

$$= \frac{u + at_1 + u + a(t_1 + t_2)}{2}$$

$$= \frac{2u + 2at_1 + at_2}{2}$$

$$V_2 = u + at_1 + \tfrac{1}{2}at_2 \longrightarrow 2$$

V_3 = average velocity t_3

$$V_3 = \frac{v_2 + v_3}{2} = \frac{2u + a(2t_1 + t_2)}{2}$$

$$= u + \tfrac{1}{2}a(2t_1 + t_2)$$

$$= u + a(t_1 + t_2) + u + a$$

$$= \frac{t_1 + t_2 + t_3}{2}$$

$$V_3 = u + at_1 + at_2 + \frac{1}{2} a t_3 \longrightarrow (3)$$

$$V_1 - V_2 = (u + \frac{1}{2} a t_1) - (u + at_1 + \frac{1}{2} a t_2)$$

$$V_1 - V_2 = -\frac{1}{2} a (t_1 + t_2) \longrightarrow (4)$$

$$V_2 - V_3 = (u + at_1 + \frac{1}{2} a t_2) - (u + at_1 + at_2 + \frac{1}{2} a t_3)$$

$$V_2 - V_3 = -\frac{1}{2} a (t_2 + t_3) \longrightarrow (5)$$

equation (4) and (5) divide

$$\frac{-\frac{a}{2}(t_1 + t_2)}{-\frac{a}{2}(t_2 + t_3)} = \frac{V_1 - V_2}{V_2 - V_3}$$

$$\frac{V_1 - V_2}{V_2 - V_3} = \frac{t_1 + t_2}{t_2 + t_3}$$

7. The two ends of a train moving with a Constant acceleration pass a Cartesian certain point with velocity u and v respectively. S.T velocity with the middle of the train passes the same point is $\sqrt{\frac{1}{2}(u^2 + v^2)}$.

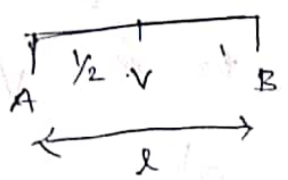
Soln:

let

v_1 - velocity the middle point of the train

l - length of the train

a - Constant acceleration



$$v^2 = u^2 + 2as$$

$$v^2 = u^2 + 2al \rightarrow (1)$$

$$v_1^2 = u^2 + 2a\, l/2$$

$$v_1^2 = u^2 + al \rightarrow (2)$$

from (1)

$$a = \frac{v^2 - u^2}{2l}$$

Sub in (2)

$$v_1^2 = u^2 + \left(\frac{v^2 - u^2}{2\cancel{l}}\right) l$$

$$= u^2 + \frac{v^2 - u^2}{2}$$

$$= \frac{2u^2 + v^2 - u^2}{2}$$

$$v_1^2 = \frac{u^2 + v^2}{2}$$

$$v_1 = \sqrt{\frac{u^2 + v^2}{2}}.$$

8. A lift ascends with a Constant acceleration 'a'. Then with a Constant velocity and finally stops with a Constant retardation 'a'. If the total distance travelled is S and the total time occupied is T. S.T the time for which the lift was ascending with Constant velocity t_2 is $\sqrt{T^2 - 4S/a}$.

Soln:

At AB:

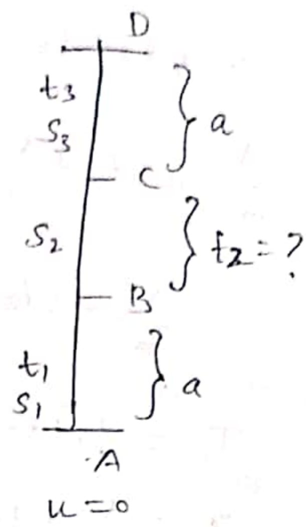
Initial velocity $= 0$.

Final velocity = v

acceleration = a

time = t_1

distance = S_1



$V = u + at$

$V = 0 + at_1$

$$\boxed{t_1 = V/a}$$

$V^2 = u^2 + 2as$

$V^2 = 0 + 2aS_1$

$$\boxed{S_1 = \frac{V^2}{2a}}$$

At BC is

$S_2 = vt_2$. [distance = speed x time]

At CD.

C is initial velocity = v

D is final velocity = 0.

acceleration = −a

distance = S_3

time = t_3

$V = u + at$

$0 = V - at_3$

$$\boxed{t_3 = V/a}$$

$V^2 = u^2 + 2as$

$0 = V^2 - 2aS_3$

$$\boxed{S_3 = V^2/2a}$$

Constant

$$S_1 + S_2 + S_3 = S$$

$$\frac{V^2}{2a} + Vt_2 + \frac{V^2}{2a} = S$$

$$\frac{V^2}{a} + Vt_2 = S \longrightarrow (1)$$

$$T = t_1 + t_2 + t_3$$

$$T = V/a + t_2 + V/a$$

$$T = \frac{2V}{a} + t_2 \longrightarrow (2)$$

Squaring,

$$T^2 = \left(\frac{2V}{a} + t_2\right)^2 \Rightarrow \frac{4V^2}{a^2} + t_2^2 + \frac{4V}{a}t_2$$

$$T^2 = \frac{4}{a}\left(\frac{V^2}{a} + Vt_2\right) + t_2^2$$

$$T^2 = \frac{4}{a}S + t_2^2$$

$$t_2^2 = T^2 - \frac{4S}{a}$$

$$t_2 = \sqrt{T^2 - \frac{4S}{a}}$$

Book work:

To find the Components of velocity and acceleration of a particle in the radial and transverse directions.

Soln:

O : pole [Constant point]

OX : Straight line (x-axis)

OY : Y - axis

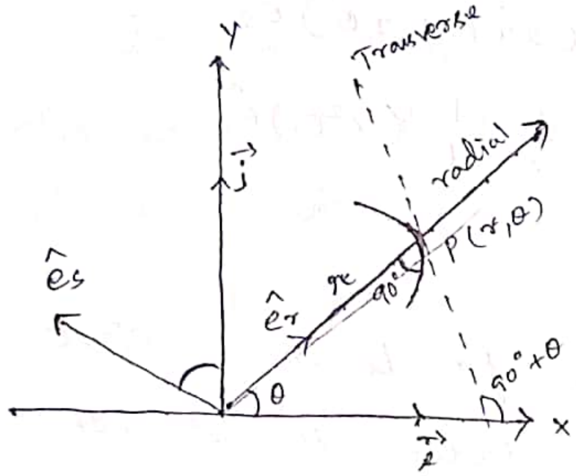
$\vec{i}, \vec{j}$: OX, OY direction

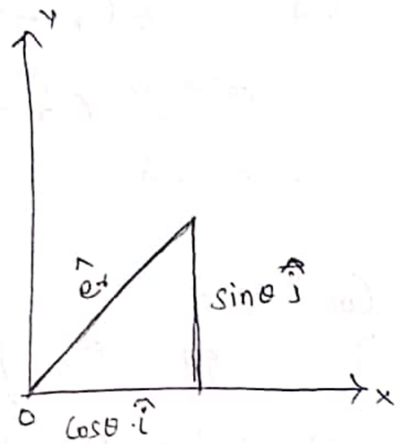
unit vector : $p(r, \theta)$ initial line

$\overrightarrow{OP} = \vec{r}$

OP in the sence in which r increases is called the radial direction and the direction perpendicular to OP in the sence in which θ increase is called the transverse direction.

Let $\hat{e_r}$ and $\hat{e_s}$ be the unit vector in these two directions.





$$\hat{e_r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{e_s} = \cos(\pi/2+\theta)\,\hat{i} + \sin(\pi/2+\theta)\,\hat{j}$$

$$\hat{e_s} = -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$$

$$d\,\hat{e_r} = (-\sin\theta\,\hat{i} + \cos\theta\,\hat{j})\,\dot{\theta}$$

$$= \dot{\theta}\,\hat{e_s}$$

$$\frac{d\hat{e_s}}{dt} = (-\cos\theta\,\hat{i} - \sin\theta\,\hat{j})\,\dot{\theta}$$

$$= -\dot{\theta}\,(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

$$= -\dot{\theta}\,\hat{e_r}$$

$$\vec{r} = r\,\hat{e_r}$$

$$\frac{d\vec{v}}{dt} = \dot{r}\,\hat{e}_r + r\,\frac{d\hat{e}_r}{dt}$$

$$v = \dot{r}\,\hat{e}_r + r\dot{\theta}\,\hat{e}_s$$

So the components of the velocity in the radial and transverse directions are $\dot{r}$ and $r\dot{\theta}$

Further the acceleration of the particle is

$$a = \frac{d\bar{v}}{dt} = \ddot{r}\,\hat{e}_r + \dot{r}\,\frac{d}{dt}\hat{e}_r + \dot{r}\dot{\theta}\,\hat{e}_s + r\ddot{\theta}\,\hat{e}_s + r\dot{\theta}\,\frac{d}{dt}\hat{e}_s$$

$$= \ddot{r}\,\hat{e}_r + \dot{r}\dot{\theta}\,\hat{e}_s + \dot{r}\dot{\theta}\,\hat{e}_s + r\ddot{\theta}\,\hat{e}_s - r\dot{\theta}\dot{\theta}\,\hat{e}_r$$

$$= (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{e}_s$$

$$= (\ddot{r} - r\dot{\theta}^2)\,\hat{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\,\hat{e}_s$$

Corollary :

If the path of the particle is a circle whose centre is o and radius is 'a' then

$$\bar{v} = (0)\,\hat{e}_r + (a\dot{\theta})\,\hat{e}_s$$

$$\bar{a} = (-a\dot{\theta}^2)\,\hat{e}_r + (a\ddot{\theta})\,\hat{e}_s$$

Because $r = a$ $\dot{r} = 0$ and $\ddot{r} = 0$. It may be noted that these results are the same as the results obtained in the previous book work.

Definition : Angular velocity

Let P be particle having a Coplaner motion. Let O be a fixed point and OA be fixed line in the plane of motion then the time-rate of change of the angle AOP is called the angular velocity of the particle about O that is if the angle AOP $= \theta$, then $\frac{d\theta}{dt} = \dot{\theta}$ is the angular velocity of the particle about O. The unit of angular velocity is one radian per Second.

In the circular motion of a particle we obtained That linear velocity V and angular velocity θ about the Centre are related as

$$v = a\dot{\theta} \quad (\text{or}) \quad \dot{\theta} = v/a$$

In general we see that θ can be obtained from the transverse $r\theta$ of the velocity by dividing it by r.

The Component of velocity of P perpendicular tob OP

$$r\theta = V\sin\phi$$

$$\theta = \frac{v\sin\phi}{r} = \frac{rv\sin\phi}{r^2}$$

$$\theta = \frac{|\vec{r}+\vec{v}|}{r^2}$$

Since $\vec{v} = \hat{v}t$

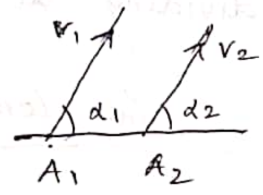
Relative angular velocity :

Let A_1 and A_2 be two particle moving in a plane. If their velocities are v_1 and v_2 Making angles α_1 and α_2 with A_1A_2 as shown in the figure. Then the Compound in the direction perpendicular to A_1 is $V_2 \sin\alpha_2 - V_1 \sin\alpha_1$ because $A/2$ relative the velocity Components of A_1 and A_2 in this direction are $V_1 \sin\alpha_1$ and $V_2 \sin\alpha_2$. So from ① of the previous Section the angular Velocity of relative to A_1 is,

$$\text{angular velocity} = \frac{V_2 \sin\alpha_2 - V_1 \sin\alpha_1}{A_1A_2} \longrightarrow ①$$

Also $\overline{A_1A_2}$ is the position vector of A_2 with reference to A_1 So if $\bar{v}_1$ and $\bar{v}_2$ are the velocity vectors of A_1 and A_2 Then the velocity of A_2 relative to A_1 is $\bar{V}_2 - \bar{V}_1$ and Consequently the angular Speed of A_2 about A_1 is obtain from ②

$$\dot{\theta} = \frac{|\overline{A_1A_2} \times (\overline{V_2 - V_1})|}{A_1A_2^2}$$



$$\dot{\theta} = \frac{|\overline{A_1A_2} \times (\overline{V_2 - V_1})|}{A_1A_2^2} \longrightarrow ②$$

Book work (concentric circles)

Two particles A_1, A_2 describe concentric circles of radii a_1, a_2 and centre O with speed v_1, v_2 we shall show that when the relative angular velocity of one particle about the other vanishes is

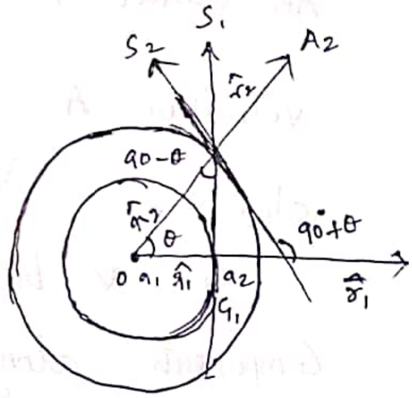
$$\cos \angle A_1OA_2 = \frac{a_1v_1 + a_2v_2}{a_2v_1 + a_1v_2}$$

Soln:

Suppose.

$$\overline{OA_1} = a_1\bar{r}_1$$
$$\overline{OA_2} = a_2\bar{r}_2$$

where $\bar{r}_1$ and $\bar{r}_2$ are unit vectors, let $\bar{s}_1$ and $\bar{s}_2$ be the unit vectors respectively perpendicular to $\bar{r}_1$ and $\bar{r}_2$ as shows that figure then the velocity of A_1 and A_2 are



$$\left| \frac{\overline{A_1A_2} \times (\bar{v}_2 - \bar{v}_1)}{A_1A_2^2} \right|$$

When this vanishes we have

$$|\overline{A_1A_2} \times (\bar{v}_2 - \bar{v}_1)| = \bar{0} \quad (\text{or})$$

$$|(\overline{OA_2} - \overline{OA_1}) \times (\bar{v}_2 - \bar{v}_1)| = 0$$

$$(a_2\bar{r}_2 - a_1\bar{r}_1) \times (v_2\bar{s}_2 - v_1\bar{s}_1) = \bar{0}$$

ie), $a_2v_2\bar{r}_2 \times \bar{s}_2 - a_2v_1\bar{r}_2 \times \bar{s}_1 - a_1v_2\bar{r}_1 \times \bar{s}_2 + a_1v_1\bar{r}_1 \times \bar{s}_1 = 0.$

If $\bar{k}$ is the unit vector perpendicular to the plane of motion so that

$$\bar{r}_1 \times \bar{s}_1 = \bar{k} \quad , \quad \bar{r}_2 \times \bar{s}_2 = \bar{k} \quad \text{then}$$

$$\bar{r}_1 \times \bar{s}_2 = \sin(90^\circ + \theta)\bar{k} = \cos\theta\,\bar{k}$$

$$\bar{r}_2 \times \bar{s}_1 = \sin(90^\circ - \theta)\bar{k} = \cos\theta\,\bar{k}$$

$$(a_2 v_2 - a_2 v_1 \cos\theta - a_1 v_2 \cos\theta + a_1 v_1)\bar{k} = 0.$$

$$a_2 v_2 - (a_2 v_1 + a_1 v_2)\cos\theta + a_1 v_1 = 0.$$

$$\cos\theta = \frac{a_1 v_1 + a_2 v_2}{a_2 v_1 + a_1 v_2}$$

Problems :

1. The line joining two points A, B is of constant length 'a' and the velocity of A, B are indirections. which makes angle α and β respectively with AB. P.T angular velocity of AB about A is $\dfrac{u\sin(\beta-\alpha)}{a\cos\beta}$ where u is the velocities A

Soln:

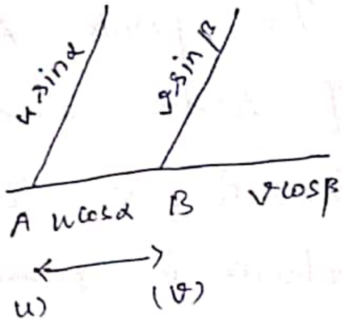
Let 'v be the velocity of B then its components along AB and perpendicular to A are $v\cos\beta$, $v\sin\beta$

for the velocity of A the components are $u\cos\alpha$, $u\sin\alpha$

Thus the components of the velocity of B relative to A are

$v\cos\beta - u\cos\alpha$,

$v\sin\beta - u\sin\alpha$



Since, AB is of Constant length, the Components of the velocity of B relative to A along AB is zero. So,

$$v\cos\beta - u\cos\alpha = 0$$

$$v = \frac{u\cos\alpha}{\cos\beta}$$

If the angular velocity of AB about A is ω

$$\omega = \frac{\text{Component of relative of B } \perp^r \text{ to AB}}{\text{length of AB}}$$

$$= \frac{v\sin\beta - u\sin\alpha}{a}$$

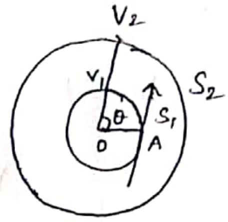
$$= \frac{\frac{u\cos\alpha}{\cos\beta}\sin\beta - u\sin\alpha}{a}$$

$$= \frac{u(\cos\alpha\sin\beta - \sin\alpha\cos\beta)}{a\cos\beta}$$

$$\omega = \frac{u\sin(\beta-\alpha)}{a\cos\beta}$$

2. Two points A_1 and A_2 describe concentre circles of radii a_1 and a_2 with speeds varying inversely as the radii, show that their relative velocity is parallel to the line A_1, A_2. when the angle between the radii through A_1 and A_2 is $\frac{2a_1a_2}{a_1^2+a_2^2}$

Soln:



O – Centre of the Circle

A_1, A_2 – Two points

a_1, a_2 - Circle radius

A_1 is

$\overline{OA_1} = a_1\hat{r}_1$ The unit vector of the forces

$\hat{S}_1 = \overline{OA_1}$ the unit vector of the perpendicular direction velocity

$$\overline{v_1} = \frac{\lambda}{a_1}\overline{S_1}$$

A_2 is

$$\overline{OA_2} = a_2\hat{r}_2$$

$S_2 = OA_2$ unit vector to the perpendicular direction velocity $\overline{V_2} = \frac{\lambda}{a_2}\overline{S_2}$

$$\overline{A_1A_2} = \overline{OA_2} - \overline{OA_1}$$
$$= a_2\hat{r}_2 - a_1\hat{r}_1$$

angular velocity $= 0$

$$\overline{A_1A_2} \times (V_2 - V_1) = 0$$

$$(a_2\bar{r}_2 - a_1\bar{r}_1) \times \left(\frac{\lambda}{a_2}\hat{s}_2 - \frac{\lambda}{a_2}\hat{s}_1 \right) = 0$$

$$\left.\begin{array}{l} a_2 \frac{\lambda}{a_2}(\bar{r}_2 \times \bar{s}_2) - a_2 \frac{\lambda}{a_1}(\bar{r}_2 \times \bar{s}_1) \\ - a_1 \frac{\lambda}{a_2}(\bar{r}_1 \times \bar{s}_2) + a_1 \frac{\lambda}{a_1}(\bar{r}_1 \times \bar{s}_1) \end{array}\right\} = 0 \longrightarrow ①$$

$$\bar{r}_1 \times \bar{s}_1 = 1.1 \sin 90^\circ \hat{k} = \hat{k}$$

$$\bar{r}_2 \times \bar{s}_2 = 1.1 \sin 90^\circ \hat{k} = \hat{k}$$

$$\bar{r}_1 \times \bar{s}_2 = 1.1 \sin(90^\circ + \theta)\hat{k} = \cos\theta \hat{k}$$

$$\bar{r}_2 \times \bar{s}_1 = 1.1 \sin(90^\circ - \theta)\hat{k} = \cos\theta \hat{k}$$

$\hat{k}$ is a unit vector

Sub the ① equation

$$\left.\begin{array}{r} a_2 \frac{\lambda}{a_2}\hat{k} - a_2 \frac{\lambda}{a_1}\cos\theta \hat{k} - a_1 \frac{\lambda}{a_2}\cos\theta \hat{k} \\ + a_1 \frac{\lambda}{a_1}\hat{k} \end{array}\right\} = 0.$$

$$\left(a_2 \frac{\lambda}{a_2} + a_1 \frac{\lambda}{a_1} \right)\hat{k} - \left(a_2 \frac{\lambda}{a_1} + a_1 \frac{\lambda}{a_2} \right)\cos\theta \hat{k} = 0$$

$$2\lambda\hat{k} - \left(\frac{a_2}{a_1} + \frac{a_1}{a_2} \right)\lambda\hat{k}\cos\theta = 0.$$

$$\lambda\hat{k}\left(2 - \left(\frac{a_2}{a_1} + \frac{a_1}{a_2} \right)\cos\theta \right) = 0.$$

$$2 - \left(\frac{a_2}{a_1} + \frac{a_1}{a_2} \right)\cos\theta = 0$$

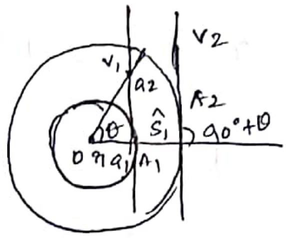
$$\left(\frac{a_2}{a_1} + \frac{a_1}{a_2} \right)\cos\theta = 2$$

$$\cos\theta = \frac{2a_1a_2}{a_1^2 + a_2^2}$$

$$\theta = \cos^{-1}\left(\frac{2a_1a_2}{a_1^2 + a_2^2} \right)$$

3. Two planets describe nearly are radii a_1 and a_2 round sun as centre with speeds varying inversely as the square roots of the radii, S.T their relative angular velocity vanishes when the angle between the radii its these planets is $\cos^{-1}\left[\frac{\sqrt{a_1 a_2}}{a_1-\sqrt{a_1 a_2}+a_2}\right]$

Soln:



O - The centre mass of the circle

A_1, A_2 - Two planets

a_1, a_2 - planets radii

A_1 is,

$OA_1 = V_1 = OA_1$ unit vector of forces

$S_1 = OA_1$. is unit vector of perpendicular direction

$$\text{velocity} = V_1 = \frac{\lambda}{\sqrt{a_1}} \bar{s}_1$$

A_2 is,

$OA_2 \rightarrow a_2$

$r_2 \rightarrow OA_2$ unit vector

$\overline{OA_2} = \overline{a_2 r_2}$

$S_2 \rightarrow OA_2$ unit vector of the perpendicular direction

$$V_2 = \frac{\lambda}{\sqrt{a_2}} \bar{S}_2$$

$$\overline{A_1A_2} = \overline{OA_2} - \overline{OA_1}$$

$$= a_2\bar{r}_2 - a_1\bar{r}_1$$

Angular velocity $= 0$

$$\overline{A_1A_2} \times (\bar{V}_2 - \bar{V}_1) = 0$$

$$(a_2\bar{r}_2 - a_1\bar{r}_1) \times \left(\frac{\lambda}{\sqrt{a_2}} \bar{s}_2 - \frac{\lambda}{\sqrt{a_1}} \bar{s}_1 \right) = 0$$

$$\left.\begin{aligned} & a_2 \frac{\lambda}{\sqrt{a_2}} (\bar{r}_2 \times \bar{s}_2) - a_2 \frac{\lambda}{\sqrt{a_1}} (\bar{r}_2 \times \bar{s}_1) \\ & - a_1 \frac{\lambda}{\sqrt{a_2}} (\bar{r}_1 \times \bar{s}_2) + a_1 \frac{\lambda}{\sqrt{a_1}} (\bar{r}_1 \times \bar{s}_1) \end{aligned}\right\} = 0 \longrightarrow (1)$$

$$\bar{r}_1 \times \bar{s}_1 = 1.1 \sin 90^\circ \hat{k} = \hat{k}$$

$$\bar{r}_2 \times \bar{s}_2 = 1.1 \sin 90^\circ \hat{k} = \hat{k}$$

$$\bar{r}_1 \times \bar{s}_2 = 1.1 \sin(90^\circ + \theta) \hat{k} = \cos\theta \hat{k}$$

$$\bar{r}_2 \times \bar{s}_1 = 1.1 \sin(90^\circ - \theta) \hat{k} = \cos\theta \hat{k}$$

$\hat{k}$ is a unit vector

Sub the (1) equation

$$\left.\begin{aligned} & a_2 \frac{\lambda}{\sqrt{a_2}} \hat{k} - a_2 \frac{\lambda}{\sqrt{a_1}} \cos\theta \hat{k} - a_1 \frac{\lambda}{\sqrt{a_2}} \cos\theta \hat{k} \\ & + a_1 \frac{\lambda}{\sqrt{a_1}} \hat{k} \end{aligned}\right\} = 0.$$

$$\sqrt{a_2}\lambda \hat{k} - \lambda \cos\theta \hat{k} \left(\frac{a_2}{\sqrt{a_1}} + \frac{a_1}{\sqrt{a_2}} \right) + \sqrt{a_1}\lambda \hat{k} = 0$$

$$\lambda \hat{k} \left[(\sqrt{a_1} + \sqrt{a_2}) - \cos\theta \left(\frac{a_2}{\sqrt{a_1}} + \frac{a_1}{\sqrt{a_2}} \right) \right] = 0.$$

$$(\sqrt{a_1} + \sqrt{a_2}) - \cos\theta \left(\frac{a_2}{\sqrt{a_1}} + \frac{a_1}{\sqrt{a_2}} \right) = 0.$$

$$\cos\theta \left(\frac{a_2}{\sqrt{a_1}} + \frac{a_1}{\sqrt{a_2}} \right) = \sqrt{a_1} + \sqrt{a_2}$$

$$\cos\theta \left[\frac{\sqrt{a_2}\, a_2 + a_1 \sqrt{a_1}}{\sqrt{a_1}\sqrt{a_2}} \right] = \sqrt{a_1} + \sqrt{a_2}$$

$$\cos\theta = \frac{(\sqrt{a_1}+\sqrt{a_2})\sqrt{a_1 a_2}}{(\sqrt{a_1})^3+(\sqrt{a_2})^3}$$

$$= \frac{\sqrt{a_1 a_2}\,(\sqrt{a_1}+\sqrt{a_2})}{(\sqrt{a_1}+\sqrt{a_2})(a_1-\sqrt{a_1 a_2}+a_2)}$$

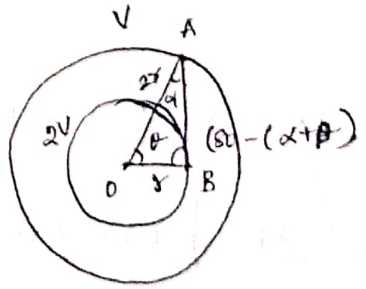
$$\cos\theta = \frac{\sqrt{a_1 a_2}}{a_1-\sqrt{a_1 a_2}+a_2}$$

$$\theta = \cos^{-1}\left(\frac{\sqrt{a_1 a_2}}{a_1-\sqrt{a_1 a_2}+a_2}\right)$$

$\therefore$ Hence proved.

4. Two points A and B move with speeds v and $2v$ along two concentric circles with centres at 'O' and radii $2r$ and r respectively. If angle $OAB = \alpha$ show that $\cot\alpha = 2$ when their relative motion is along AB.

Soln:



$$\cos\theta = \frac{a_1 v_1 + a_2 v_2}{a_2 v_1 + a_1 v_2}$$

$a_1 = 2r, \quad a_2 = r$

$v_1 = v, \quad v_2 = 2v$

$$\cos\theta = \frac{2rv + r2v}{2r(2v)+r(r)}$$

$$= \frac{4rv}{5rv}$$

$$\cos\theta = 4/5$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$= 1 - \frac{16}{25}$$

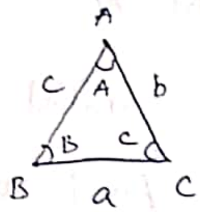
$$= \frac{25-16}{25}$$

$$\sin^2\theta = 9/25$$

$$\sin\theta = 3/5$$

By sine formula

$$\frac{r}{\sin \angle OAB} = \frac{2r}{\sin \angle ABO}$$



$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$\frac{r}{\sin\alpha} = \frac{2r}{\sin(180^\circ - (\theta+\alpha))}$$

$$\frac{r}{\sin\alpha} = \frac{2r}{\sin(\theta+\alpha)}$$

$$\sin(\theta+\alpha) = 2\sin\alpha$$

$$\sin\alpha = \frac{\sin(\theta+\alpha)}{2}$$

$$2\sin\alpha = \sin\theta\cos\alpha + \cos\theta\sin\alpha$$

$$2\sin\alpha = 3/5\cos\alpha + 4/5\sin\alpha$$

$$2\sin\alpha - 4/5\sin\alpha = 3/5\cos\alpha$$

$$\sin\alpha\left(2 - 4/5\right) = 3/5\cos\alpha$$

$$\sin\alpha\left(6/5\right) = 3/5\cos\alpha$$

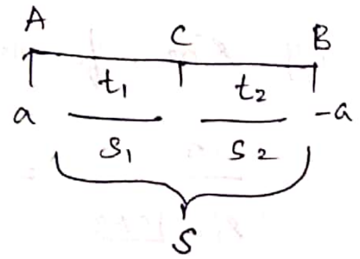
$$\frac{\cos\alpha}{\sin\alpha} = 2$$

$$\cot\alpha = 2$$

5. A body travels a distances S in t seconds. If starts from rest and ends at rest in the first part of the journey, it moves with a Constant acceleration 'a' and in the second part with a Constant retardation 'a''. S.T

$$aa't^2 = 2S(a+a') \quad (or) \quad t^2 = \frac{2S(a+a')}{aa'}$$

Soln:



At AC:

initial velocity $= 0$

Final velocity $= V$

time $= t_1$

distance $= S_1$

acceleration $= a$.

W.K.T

$V = u + at$

$V = 0 + at_1$

$$\boxed{t_1 = V/a}$$

$V^2 = u^2 + 2aS$

$V^2 = 0 + 2aS_1$

$$\boxed{S_1 = \frac{V^2}{2a}}$$

At CB:

Initial velocity $= V$

Final velocity $= 0$.

time $= t_2$

distance $= s_2$

acceleration $= -a'$

$$v = u + at$$

$$0 = v - a't_2$$

$$\boxed{t_2 = v/a'}$$

$$v^2 = u^2 + 2as$$

$$0 = v^2 - 2a's_2$$

$$\boxed{s_2 = \frac{v^2}{2a'}}$$

$$t = t_1 + t_2$$

$$= \frac{v}{a} + \frac{v}{a'}$$

$$= v\left(\frac{1}{a} + \frac{1}{a'}\right)$$

$$t = v\left(\frac{a'+a}{aa'}\right)$$

$$t^2 = v^2\left(\frac{a'+a}{aa'}\right) \longrightarrow \text{(I)}$$

$$s = s_1 + s_2$$

$$= \frac{v^2}{2a} + \frac{v^2}{2a'}$$

$$= \frac{v^2}{2}\left(\frac{1}{a} + \frac{1}{a'}\right)$$

$$s = \frac{v^2}{2}\left(\frac{a'+a}{aa'}\right)$$

$$v^2 = \frac{2s\,(aa')}{(a'+a)}$$

Sub in 1

$$t^2 = \frac{2s\,(aa')}{(a'+a)}\left(\frac{a'+a}{aa'}\right)^2$$

$$t^2 = \frac{2s\,(a'+a)}{aa'}$$